Amplification of a magnetic field in systems with a finite electric conductivity

Yu. Dolinsky* and T. Elperin†
The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel
(Received 5 October 1999)

In this study we investigated the kinematics of electromagnetic fields with various configurations of the magnetic and electric fields in systems with spherical and cylindrical symmetry using the exact solutions to the diffusion equation for the electromagnetic field in systems with moving boundaries. In the class of self-similar solutions we determined analytically conditions for the amplification of the electromagnetic field as functions of material velocity and electric conductivity. We investigated also the effects of the amplification of an electromagnetic field during propagation of a spherical shock wave in a dielectric material whereby the shock wave causes transformation of a material into a conductor.

PACS number(s): 03.50.De, 41.20.Gz

I. INTRODUCTION

One of the most efficient methods for generating strong and ultrastrong magnetic fields is the compression of magnetic flux in conducting systems. Examples of such systems are conducting metallic shells (see, e.g., Ref. [1] and references therein), conducting liquids or strongly ionized mediums, e.g., plasma produced after the shock wave front (see, e.g., Refs. [2,3]). A mechanism of the amplification of a magnetic field is associated with a negative work performed by ponderomotive forces. Thus, the necessary condition for an amplification of a magnetic field reads (for details see Refs. [4–6]):

$$\int \frac{\vec{u}}{c} \cdot (\vec{j} \times \vec{H}) d\vec{r} < 0, \tag{1}$$

where $\vec{u}(\vec{r},t)$ is a local velocity of a conducting medium, $\vec{j}(\vec{r},t)$ is a density of an electric current, and $\vec{H}(\vec{r},t)$ is a magnetic field.

Although the physical nature of these phenomena is quite transparent, it is not always easy to determine the conditions for the amplification of a magnetic field. The reason is that the boundary conditions together with the symmetry of an electromagnetic field yield certain kinematic restrictions. Condition (1) is not always satisfied for a given configuration of an electromagnetic field, e.g., it is not valid in a case of an imploding conducting cylinder with electric current distributed across the whole cross section and directed along the axis of the cylinder. For an electric current with an azimuthal symmetry $\partial j_z/\partial\varphi = 0$ where φ is an azimuthal coordinate, this conclusion is quite evident. Indeed, it follows from the effect of z-pinch of plasma or of a cylindrical conductor (see, e.g., Ref. [7]) whereby ponderomotive forces act to compress a conductor and, therefore, perform a positive work during its implosion. In a case without an azimuthal symmetry, condition (1) does not allow to predetermine the feasibility of generating a magnetic field.

In systems with a finite electric conductivity, condition (1) is not sufficient since in this case the power of ponderomotive forces must be larger than a joule dissipation rate (for details see Ref. [7]). Another problem arising when investigating the feasibility of generating a magnetic field is that in many cases a compression of a magnetic flux is caused by a shock wave which transforms a material into a conducting state (see, e.g., Refs. [2,3]). Here before the shock wave front a material is dielectric while after the shock wave front it becomes a conductor. The velocity of a shock wave differs from the material velocity while the power of the pondermotive forces is determined by a material velocity and not by the velocity of a shock wave. Therefore, the condition for an amplification of a magnetic field under a given shock wave velocity implies a certain relation between the velocity of a shock wave and a material velocity.

Since theoretical and experimental studies on generating strong and ultrastrong magnetic fields by compressing magnetic flux inside have been performed for many years, there is a large number of publications in this field (see, e.g., Refs. [1,8] and references therein). However, most of the theoretical studies employ electrotechnical approximations or simplified electrodynamic schemes which are not satisfactory as far as a self-consistent solution of the corresponding boundary value problem for the electromagnetic fields is concerned (see, e.g., Refs. [9,10]).

Among the studies where this problem is considered self-consistently, we must mention the studies in Refs. [11,12]. The study in Ref. [12] investigates the dynamics of an azimuthal magnetic field H_φ in an ideally conducting plasma with z-pinch symmetry while in Ref. [11] a finite electric conductivity plasma with the geometry of a θ-pinch is analyzed. In both cases plasma occupies a cylindrical domain $\rho < \overline{\rho}(t)$ [$\overline{\rho}(t)$ is a boundary of a cylindrical domain] and its electric conductivity is determined by a temperature with a self-similar distribution. On the other hand, for practical applications it is of interest to consider an amplification of a magnetic field when plasma occupies a domain $\rho > \overline{\rho}(t)$ while a region $\rho < \overline{\rho}(t)$ is occupied by a stationary dielectric material. Solution of the latter problem will allow us to answer a number of questions, e.g., the dependence of the con-

*Electronic address: yuli@menix.bgu.ac.il
†Electronic address: elperin@menix.bgu.ac.il

ditions for the amplification of a magnetic field on the electric conductivity of plasma, on the velocity of a jump of electric conductivity, on the fluid velocity at the moving front of the electric conductivity jump, and on the symmetry of a magnetic field. On the other hand, an exact analytical solution to the problem of an amplification of magnetic fields in systems with distributed parameters can be obtained only in the class of self-similar solutions. In the case of the system with a finite electric conductivity the latter implies the constant magnetic Reynolds number (see Refs. [4,5,11,12]). Note that when in a problem with moving boundaries a condition for self-similarity is not satisfied, evolution of such a system is accompanied by variation of spatial and time scales which renders a very involved description of the system.

In this study we still remain in the framework of the self-similar problem but consider a broader class of the magnetic fields and take into account the difference between the velocity of the front of an electric conductivity jump and a fluid velocity at this front. We will restrict ourselves to a kinematic level of description whereby thermodynamic processes associated with the variation of temperature, magnetic, and thermodynamic pressures are not considered. The main goal of the present study is to determine the thresholds for a fluid velocity and a velocity of the front of electric conductivity which are required for the amplification of the electromagnetic field with different configurations in systems with different geometries.

II. FORMULATION OF THE PROBLEM

The general scheme of the solution of the problem described above is as follows. Let a surface $F(x,y,z,t)=0$ separate a region occupied by a conducting material from a region with an electric conductivity $\sigma=0$. Let $\vec{v}_F$ be a velocity of a front $F(x,y,z,t)=0$ which is determined by the equation

$$\frac{\partial F}{\partial t}+\vec{v}_F\cdot\vec{\nabla}F=0,$$

and $\vec{u}$ is a material velocity at the front $\vec{u}=\vec{u}(\vec{r},t)|_{F(x,y,z,t)=0}$. Then with the accuracy of the order of $\max\{v_F^2/c^2,u^2/c^2\}$ we can write the equation for a magnetic $\vec{H}$ and electric $\vec{E}$ fields in the domain with $\sigma=0$ as follows:

$$\vec{\nabla}^2\vec{H}=0. \tag{2}$$

Inside the domain occupied by a conducting material instead of Eq. (2) we have

$$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\vec{j},\quad \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t}. \tag{3}$$

Equations (2,3) must be supplemented with the Ohm's law for a moving medium which in the framework of the simple magnetohydrodynamic model can be written as follows (see, e.g., Ref. [7]):

$$\vec{j}=\sigma\left[\vec{E}+\frac{1}{c}(\vec{u}\times\vec{H})\right]. \tag{4}$$

Hereafter we will consider only solenoidal fields, i.e., the following conditions are satisfied:

$$\vec{\nabla}\cdot E=0,\qquad \vec{\nabla}\cdot\vec{H}=0. \tag{5}$$

Equations (2,3) must be supplemented with the boundary conditions which for the systems with a finite electric conductivity reduce to the conditions of continuity of electric and magnetic fields at the surface $F(x,y,z,t)=0$ and requirement for the absence of the external sources of electromagnetic field at infinity:

$$\vec{E}_+=\vec{E}_-,\vec{H}_+=\vec{H}_-,E(\vec{r},t)\propto\frac{1}{r^{\alpha}},\qquad H(\vec{r},t)\propto\frac{1}{r^{\beta}}\quad\text{at}\quad r\to\infty,\alpha,\beta\geqslant 1, \tag{6}$$

where subscripts + and − denote values at different sides of the surface. The latter two conditions follow from the requirement for a vanishing flux of the electromagnetic energy at infinity.

Equations (2,4) together with boundary conditions (6) and the condition of self-similarity of a solution, which will be specified below, completely determine (with an accuracy of a constant multiplier) a set of possible self-similar solutions. A boundary condition (6) yields a transcendental equation which allows to determine a condition for amplification of an electromagnetic field for a given configuration of the electromagnetic field and geometry of the system.

III. AMPLIFICATION OF MAGNETIC FIELDS IN CONFIGURATIONS WITH A CYLINDRICAL SYMMETRY

Consider a case with a cylindrical symmetry. Amplification of a magnetic field in systems with a cylindrical symmetry was investigated in the above mentioned studies [11], [12] and also in Refs. [4,5,13]. In Ref. [13] we considered an infinitely long ideal conducting shell moving with a constant radial velocity $\dot{\rho}(t)$.

For the completeness of the exposition in the following we present briefly the results obtained in Ref. [13]. In the latter study we had considered a case when a magnetic field has only z-component H_z directed along the axis of symmetry and the electric field has only an azimuthal component E_φ. In physics of plasmas such a configuration is called θ-pinch symmetry (see Ref. [7], Chap. 8, Sec. 68). At the surface of an ideally conducting shell the first two conditions in Eq. (6) must be replaced by another condition

$$E_\varphi-\beta H_z=0,\quad \beta=\frac{\dot{\rho}}{c}. \tag{7}$$

The latter condition follows from Eq. (4) at $\sigma\to\infty$.

The fields E_φ and H_z in Ref. [13] are determined from the following relations:

$$E_\varphi = -\frac{1}{c}\frac{\partial A_\varphi}{\partial t}, \quad H_z = \frac{\partial A_\varphi}{\partial \rho} + \frac{A_\varphi}{\rho}, \tag{8}$$

where a formula for a vector potential reads

$$A_\varphi = a_0 \left(\frac{\overline{\rho(t)}}{\rho_0} \right)^{-1} \Phi_\varphi \left(\frac{\rho}{\overline{\rho(t)}} \right), \tag{9}$$

where a_0 is a normalization constant, $\overline{\rho(t)}$ and ρ_0 are an instantaneous and the initial radii of the ideally conducting shell,

$$\Phi_\varphi = \chi F\left(\frac{3}{2},1;2;\beta^2\chi^2\right), \quad \chi = \frac{\rho}{\overline{\rho(t)}},$$

and $F(\overline{\alpha},\overline{\beta};\overline{\gamma};z)$ is a hypergeometric function (see, e.g., Ref. [14]). Using Eqs. (8,9) electric E_φ and magnetic H_z fields can be presented as follows:

$$E_\varphi = \beta \chi H_z,$$

$$H_z = \frac{\Phi_0}{2\pi I_0 \rho^2(t)} \left[F\left(\frac{3}{2},1;2;\beta^2\chi^2\right) + \frac{3}{2}\beta^2\chi^2 F\left(\frac{5}{2},2;3;\beta^2\chi^2\right) \right], \tag{10}$$

where Φ_0 is magnetic flux which remains constant in the case of compression by an ideally conducting shell (see, e.g., Ref. [1]). I_0 is a constant which is determined by the following formula:

$$I_0 = \int_0^1 \chi \, d\chi \left[F\left(\frac{3}{2},1;2;\beta^2\chi^2\right) + \frac{3}{2}\beta^2\chi^2 F\left(\frac{5}{2},2;3;\beta^2\chi^2\right) \right].$$

The latter expression can be easily calculated as a power series of the parameter β^2. In the zeroth approximation in the parameter β^2 magnetic field H_z remains homogeneous and $A_\varphi = [\Phi_0/2\pi\rho(t)]\chi$. The above solution describes the amplification of a magnetic field in the cylindrical cavity with ideally conducting walls which implodes with a constant velocity. These ''walls'' can be of a different nature, e.g., a shock wave front. The latter case is considered in the present study. Since the approximation of an ideal conductor implies that the dissipation processes are neglected in this study we considered a more general case with a finite electric conductivity.

Taking into account the above results [see Eq. (10)] consider now a system with a finite electric conductivity. First we will study a case when a cylinder $\rho < \overline{\rho(t)}$ is occupied by a stationary dielectric material and a space outside the cylinder $\rho > \overline{\rho(t)}$ is occupied by a conducting material, e.g., plasma or conducting liquid, and $\overline{\rho(t)}$ is the location of a shock wave front. Under the effect of the shock wave which moves with a speed $\dot{\overline{\rho}}(t)$ a material is transformed into a conducting state due to ionization. Material velocity after the front of a shock wave $u(\rho,t) = u_0(t)\overline{u(\chi)}$ where $u_0(t)$ is material velocity at the wave front, i.e., $\overline{u(1)} = 1$ and before the front $u = 0$. Note that $u_0(t) \equiv u(\overline{\rho(t)},t) \neq \dot{\overline{\rho}}(t)$.

Let us write A_φ as follows:

$$A_\varphi = \left(\frac{\overline{\rho(t)}}{\rho_0} \right)^{\gamma} \Phi_\varphi \left(\frac{\rho}{\overline{\rho(t)}} \right) \tag{11}$$

and for Φ_φ using Eqs. (3), (4), and (8) we obtain the following equation:

$$\frac{\partial^2 \Phi_\varphi}{\partial \chi^2} + \frac{1}{\chi}\frac{\partial \Phi_\varphi}{\partial \chi} - \frac{\Phi_\varphi}{\chi^2} = \nu(\chi) \left[\left(\gamma + \frac{D\overline{u(\chi)}}{\chi} \right) \Phi_\varphi + [D\overline{u(\chi)} - \chi] \frac{\partial \Phi_\varphi}{\partial \chi} \right]. \tag{12}$$

Here $\nu(\chi) = (4\pi/c^2)\overline{\sigma(\chi)}\dot{\rho}(t)\overline{\rho(t)}\sigma_0(t)$, $D = [u_0(t)/\dot{\overline{\rho}}(t)]$.

Equation (12) was derived under an assumption that a magnetic Reynolds number for a front velocity is constant, i.e.,

$$\frac{4\pi}{c^2}\dot{\overline{\rho}}(t)\overline{\rho(t)}\sigma_0(t) = \text{const} \equiv \nu_0, \tag{13}$$

and electric conductivity is given by $\sigma(\rho,t) = \sigma_0(t)\overline{\sigma(\chi)}$.

Equation (13) implies the existence of a self-similar solution. Certainly the feasibility of such solution depends upon various physical parameters. When the behavior of the system is determined mainly by a motion of the boundary separating between the regions with different electric conductivity the solution at enough large times approaches the self-similar solution with magnetic Reynolds number ν_0. It is conceivable to suggest that in a more general case the system can be described by matched self-similar solutions with different magnetic Reynolds numbers at different time intervals. In addition, the self-similar solutions allow us to determine the main parameters affecting the behavior of a system and validate the numerical solutions of the governing equations [Eqs. (2)–(6) in this study].

Consider now a case when after the shock wave front a material becomes incompressible, i.e., the sound velocity after the shock wave front is much larger than a material velocity. Assume that the ratio D of the material velocity at the wave front to the velocity of the wave does not change with time and that the temperature after the wave front does not change. The essence of the latter assumptions is that the variations of the parameters after the shock wave front are small compared with the jump of the thermodynamic parameters at the shock wave front. This assumption and the previous assumption about the incompressibility of the material after the shock wave front are valid for strong shock waves. Therefore $\overline{\sigma(\chi)} = 1$ and for the incompressible material $(\vec{\nabla} \cdot \vec{u} = 0)$ after the wave front $\overline{u(\chi)} = 1/\chi$. The solution of Eq. (12) which tends to zero when $\chi \to \infty$ can be written as follows:

$$\Phi_\varphi(\chi) = \frac{a_0}{\chi} \Psi\left(-\frac{\gamma+1}{2}, -\frac{\nu_0 D}{2}; -\frac{\nu_0 \chi^2}{2} \right), \tag{14}$$

where $\Psi(\overline{a},\overline{c};z)$ can be expressed through Kummer's functions (see, e.g., Ref. [15]):

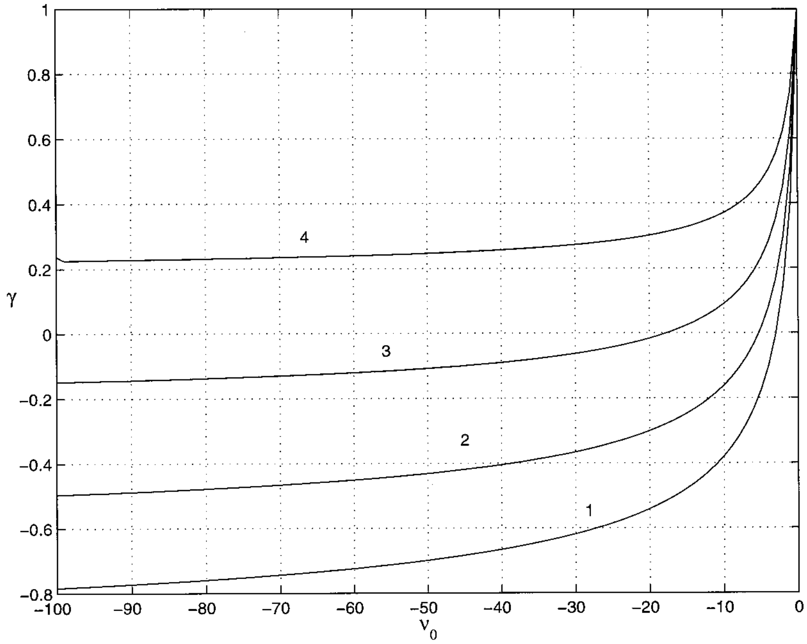

FIG. 1. The dependence of the exponent γ in Eq. (18) vs magnetic Reynolds number ν_0 for various values of the ratio of a fluid velocity at the front to a front velocity D. $1-D=1;2-D=0.8;3-D=0.6;4-D=0.4$.

$$\Psi(\overline{a},\overline{c};z)=\frac{\Gamma(1-\overline{c})}{\Gamma(\overline{a}-\overline{c}+1)}\overline{\Phi}(\overline{a},\overline{c};z)+\frac{\Gamma(\overline{c}-1)}{\Gamma(\overline{a})}z^{1-\overline{c}}\overline{\Phi}(\overline{a}-\overline{c}+1,2-\overline{c};z). \quad (15)$$

$\Gamma(z)$ is the gamma function and for $\overline{c}\neq 0,-1,-2,\ldots$ Kummers's functions can be written as power series

$$\overline{\Phi}(\overline{a},\overline{c};z)=\sum_{n=0}^{\infty}\alpha_n z^n, \quad \alpha_0=1, \quad \alpha_n=\frac{\alpha_{n-1}}{n}\left(\frac{\overline{a}+n-1}{\overline{c}+n-1}\right),$$

$$n=1,2,\cdots. \quad (16)$$

For $|z|\gg|\overline{a}|,|\overline{c}|$, function $\Psi(\overline{a},\overline{c},z)$ has the following asymptotic behavior (see, e.g., Ref. [15]):

$$\Psi(\overline{a},\overline{c};z)\sim\frac{1}{z^{\overline{a}}}. \quad (17)$$

Thus according to Eq. (14) $\Phi_\varphi(\chi)\propto\chi^\gamma$. Therefore, the requirement that a vector potential vanishes for $\chi\to\infty$, yields a condition $\gamma<0$. For $\gamma>0$ the electromagnetic field does not vanish at $\chi\to\infty$.

In the region $\rho<\rho(t)$, a vector potential A_φ is determined by Eqs. (2,8) and can be represented as follows:

$$A_\varphi=a_0\left(\frac{\overline{\rho}(t)}{\rho_0}\right)^\gamma\chi. \quad (18)$$

The continuity of electric and magnetic fields at the boundary $\chi=1$ [Eq. (6)] reduces to the condition of the continuity of the function $(\partial/\partial\chi)(\log A_\varphi)$ and yields the following transcendental equation which determines a dependence of the parameter γ on ν_0 and D:

$$\overline{a}\Psi\left(\overline{a}+1,\overline{c}+1,-\frac{\nu_0}{2}\right)=\frac{2}{\nu_0}\Psi\left(\overline{a},\overline{c},-\frac{\nu_0}{2}\right),$$

$$\overline{a}=-\frac{\gamma+1}{2}, \quad \overline{c}=-\frac{\nu_0 D}{2}. \quad (19)$$

The dependence of the exponent γ vs a magnetic Reynolds number ν_0 and D obtained from the numerical solution of Eq. (19) is shown in Fig. 1. Intersection of the curve with ν_0 axis determines the threshold value of ν_0 which is required for an amplification of a magnetic field, i.e., an increase of the total energy of a magnetic field in the domain $\rho<\overline{\rho}(t)$. At the threshold value of ν_0 the rate of a mechanical work performed against the ponderomotive forces equals the joule dissipation rate. Since the rate of a mechanical work performed against ponderomotive forces is determined by a fluid velocity and not by the propagation speed of a front, a parameter which determines this work is $\nu_0 D$. For small values of D (see Fig. 1, curve 4), amplification of a magnetic field does not occur even at $\nu_0\to\infty$, i.e., propagation of the electric conductivity front without a fluid flow is not accompanied by the amplification of a magnetic field.

Another case of an amplification of a magnetic field that can be realized with the same symmetry of an electromagnetic field which is determined by Eqs. (8,11) is when a conducting fluid (e.g., plasma) occupies an internal region $\rho<\overline{\rho}(t)$. In this case in the internal region an electromagnetic field is determined by Eqs. (3,4) and in the external region one can assume $H=0$ similar to the case of an infinite solenoid. Thus the boundary condition in this case reads:

$$H_z(\overline{\rho}(t),t)=0. \quad (20)$$

Under the assumptions that the temperature dependence of the electric conductivity can be described by Spitzer's formula $\sigma\propto T^{3/2}$, such a problem was studied in Ref. [12].

For comparison consider here a case when an electric conductivity during implosion does not change and plasma density $\mu(t)$ is spatially homogeneous. Using a continuity equation $\dot{\mu}+\vec{\nabla}\cdot(\mu\vec{u})=0$ we find that $\overline{u}(\chi)=\chi$, $\mu(t)=\mu_0\rho_0^2/\overline{\rho}^2(t)$, where μ_0 is density at $t=0$. Then Eq. (12) yields

$$\frac{\partial^2\Phi_\varphi}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial\Phi_\varphi}{\partial\chi}-\frac{\Phi_\varphi}{\chi^2}=\nu_0(\gamma+1)\Phi_\varphi. \tag{21}$$

In deriving Eq. (21) we assumed for simplicity that the velocity of an electric conductivity jump and a velocity of fluid at the front are equal, i.e., $D=1$. Nonsingular at $\chi=0$ solution of Eq. (21) is given by a Bessel function of the first order (see, e.g., Ref. [14]):

$$\Phi_\varphi(\chi)=J_1(\alpha\chi),\quad \alpha^2=-\nu_0(\gamma+1). \tag{22}$$

A magnetic field according to Eq. (8) is given by the following formula:

$$H_z(\chi)=a_0\left(\frac{\overline{\rho}(t)}{\rho_0}\right)^\gamma\frac{\alpha}{\overline{\rho}(t)}J_0(\alpha\chi) \tag{23}$$

and electric field

$$E_\varphi(\chi)=\beta a_0\left(\frac{\overline{\rho}(t)}{\rho_0}\right)^\gamma\frac{1}{\rho(t)}\left[\frac{\alpha^2}{\nu_0}J_1(\alpha\chi)+\alpha\chi J_0(\alpha\chi)\right], \tag{24}$$

where J_0 is a Bessel function of zero order. A requirement [Eq. (20)] $H_z(1)=0$ according to Eq. (23) can be met only for real values of the argument of a Bessel function, and thus, $\alpha^2>0$. Let α_k be roots of equation $J_0(\alpha)=0$. Then according to Eq. (22) a growth rate of a magnetic field is given by

$$\gamma=-\frac{\alpha_k^2}{\nu_0}-1. \tag{25}$$

Note that when $\nu_0>0$, $\overline{\rho}(t)/\rho_0>1$, and Eqs. (23,24) show that $\gamma>0$ corresponds to amplification of the electromagnetic field while for $\nu_0<0$, $\overline{\rho}(t)/\rho_0<1$, and amplification of the electromagnetic field corresponds to $\gamma<0$.

Then Eq. (25) shows that there exist three characteristic regions of parameter ν_0. A region $\nu_0>0$ corresponds to an expansion, and since then $\gamma<0$ amplification does not occur. In the region $-\alpha_k^2<\nu_0<0$ according to Eq. (25) $\gamma>0$, and there is also no amplification. Thus an amplification of a magnetic field occurs only in the region $\nu_0<-\alpha_k^2$ (note that a minimum value of a root $\alpha_0\sim 2.401$). In this region the rate of the work performed against the ponderomotive forces is sufficiently large to compensate joule losses.

IV. AMPLIFICATION OF A MAGNETIC FIELD DEPENDING ON AZIMUTHAL COORDINATE

Another configuration of a magnetic field which can be realized in systems with a cylindrical symmetry is when an electric field is directed along the z axis (z-pinch symmetry). Two cases must be considered here. The first case is when an electric field does not depend on φ, and when there is only one nonzero component of a magnetic field H_φ. The second case is when an electric field does not have an azimuthal symmetry and there are two components, H_ρ and H_φ. Amplification of a magnetic field in a configuration with azimuthal symmetry of electric field was investigated in Ref. [4] where a condition for a spontaneous excitation of a magnetic field was determined. In a linear problem a condition for an excitation of a magnetic field does not depend upon the magnitude of a magnetic field and, therefore, a condition for a spontaneous excitation of a magnetic field and a condition for an amplification of a magnetic field are equivalent. In Ref. [4] it was shown that an amplification of a magnetic field occurs not during implosion of a cylindrical conductor but during its expansion. The physical mechanism for such behavior is as follows. When a conducting material occupies a region $\rho<\overline{\rho}(t)$, and electric current inside it is directed along the z axis, ponderomotive forces act to compress a conductor. Since for an amplification of a magnetic field a work must be performed against ponderomotive forces, an amplification of a magnetic field occurs during expansion of a conductor.

Consider now an amplification of a magnetic field in configurations with azimuthal dependence of an electric field. When in systems with z-pinch, the electric field depends upon the azimuthal coordinate, i.e., when $\partial E_z/\partial\varphi\neq 0$, a magnetic field in addition to a component H_φ has also a radial component H_ρ. In Ref. [13] a configuration of such an electromagnetic field was determined and its dynamics inside an imploding ideally conducting shell was analyzed. The obtained results are briefly summarized in the following. An expression for a vector potential $\vec{A}=(0,0,A_z)$ reads

$$A_z=a_0\exp(im\varphi)\Phi_m\left(\frac{\rho}{\overline{\rho}(t)}\right),\quad \rho<\overline{\rho}(t),$$
$$\Phi_m\left(\frac{\rho}{\overline{\rho}(t)}\right)=\chi^m F\left(\frac{m+1}{2},\frac{m}{2};m+1;\beta^2\chi^2\right),\quad \chi=\frac{\rho}{\overline{\rho}(t)}, \tag{26}$$

where as previously $\beta=\dot{\overline{\rho}}(t)/c$ ($\dot{\overline{\rho}}(t)$ is an implosion velocity) and $F(\overline{\alpha},\overline{\beta};\overline{\gamma};z)$ is a hypergeometric function. Solution (26) satisfies boundary condition $\gamma\Phi_m(1)=0$ which can be derived from $E_z+\dot{\overline{\rho}}(t)/cH_\varphi=0$ [compare with condition (7)]. Magnetic fields H_ρ and H_φ are determined by the equations

$$H_\rho=\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi},\quad H_\varphi=-\frac{\partial A_z}{\partial\rho}, \tag{27}$$

and electric field E_z by Eq. (8). Thus we find that

$$H_\rho=-\frac{a_0 m\chi^{m-1}}{\overline{\rho}(t)}F\left(\frac{m+1}{2},\frac{m}{2};m+1;\beta^2\chi^2\right)\sin(m\varphi),$$

$$H_\varphi=\frac{a_0 m\chi^{m-1}}{\overline{\rho}(t)}\left[A(\chi)+\frac{\beta^2\chi^2}{2}B(\chi)\right]\cos(m\varphi),$$

$$E_z=-\beta\chi H_\varphi\cos(m\varphi),$$

where

$$A(\chi)=F\left(\frac{m+1}{2},\frac{m}{2};m+1;\beta^2\chi^2\right),$$

$$B(\chi)=F\left(\frac{m+3}{2},\frac{m+2}{2};m+2;\beta^2\chi^2\right).$$

Now consider a case with a finite electric conductivity. Assume first that a region inside a cavity $\rho<\overline{\rho(t)}$ is occupied by a stationary dielectric while a space outside the cavity $\rho>\overline{\rho(t)}$ is occupied by a conducting material. Then vector potential inside the cavity is determined by Eq. (2) and outside the cavity by Eqs. (3)–(5). In both regions it can be written as follows:

$$A_z=a_0\left(\frac{\rho(t)}{\rho_0}\right)^{\gamma}\exp(im\varphi)\Phi_m\left(\frac{\rho}{\rho(t)}\right). \tag{28}$$

Eqs. (3)–(5) yield an equation for $\Phi_m(\chi)$. In the case of an incompressible conducting material, the equation of continuity ($\vec{\nabla}\cdot\vec{u}=0$) yields $\overline{u(\chi)}=1/\chi$ and

$$\frac{\partial^2\Phi_m}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial\Phi_m}{\partial\chi}-\frac{m^2}{\chi^2}\Phi_m=\nu_0\left[\gamma\Phi_m+\left(\frac{D}{\chi}-\chi\right)\frac{\partial\Phi_m}{\partial\chi}\right]. \tag{29}$$

Solution of the latter equation vanishing at $\chi\rightarrow\infty$ can be written as follows:

$$\Phi_m(\chi)=\chi^s\Psi\left(\overline{a},\overline{c};-\frac{\nu_0\chi^2}{2}\right) \tag{30}$$

where

$$\overline{a}=\frac{s-\gamma}{2},\quad \overline{c}=1-\sqrt{m^2+\left(\frac{\nu_0 D}{2}\right)^2},$$

$$s=\frac{\nu D}{2}-\sqrt{m^2+\left(\frac{\nu_0 D}{2}\right)^2}$$

and $\Psi(\overline{a},\overline{c},z)$ is determined by Eq. (15).

Taking into account an asymptotic behavior (17) it is easily seen that $\Phi_m(\chi)\rightarrow 0$ when $\chi\rightarrow\infty$, provided that $\gamma<0$. This condition implies that the self-similar solution which satisfies the boundary conditions exists only in the case of an expanding dielectric domain $[\rho<\overline{\rho(t)},\dot{\overline{\rho(t)}}>0]$. In the latter case a condition $\gamma<0$ corresponds to the attenuation of electromagnetic field. The existence of the self-similar solution for $\dot{\overline{\rho(t)}}<0$ and $\gamma<0$ is impossible. Indeed, its existence would imply that in the case with a finite conductivity (with joule losses) the vector potential A_z grows faster than in the case of the ideal conductor where according to Eq. (26) $\gamma=0$. Since in this case a transcendental equation for a growth rate does not allow to obtain the analytical dependence $\gamma=\varphi(\nu_0)$, it is not presented here.

Consider now another case when a field depends upon azimuthal coordinate, and a conducting material occupies a region $\rho<\overline{\rho(t)}$. In this case the analytical dependence $\gamma=\varphi(\nu_0)$ can be obtained. Skipping a case with $D\neq 1$, consider a solution when a fluid velocity at the front of an electric conductivity jump and a speed of the front propagation are equal. Similar to the previous analysis assume that a solution for A_z is given by Eq. (28), density of fluid is spatially homogeneous and an electric conductivity of fluid does not change during implosion. Then Eq. (29) becomes $[\overline{u(\chi)}=\chi,\ D=1]$:

$$\frac{\partial^2\Phi_m}{\partial\chi^2}+\frac{1}{\chi}\frac{\partial\Phi_m}{\partial\chi}-\frac{m^2}{\chi^2}\Phi_m=\nu_0\gamma\Phi_m. \tag{31}$$

Nonsingular at $\chi=0$ solution of Eq. (21) is $\Phi_m=J_m(\alpha\chi)$, where $J_m(z)$ is a Bessel function of the mth order, $\alpha^2=-\nu\gamma$ and $m=1,2\ldots$. A condition $\Phi_m(\chi)\rightarrow 0$ at $\chi\rightarrow\infty$ yields that parameter α is real and therefore $\nu\gamma<0$.

In the region $\rho>\overline{\rho(t)}$ a vector potential $A_z=a_0\exp(im\varphi)\chi^{-m}[\rho(t)/\rho_0]^{\gamma}$.

Continuity of electric and magnetic fields yields a condition

$$\left.\frac{\partial J_m}{\partial\chi}\right|_{\chi=1}=-mJ_m(\alpha).$$

For a given value of the root of the latter equation α_m, $\gamma=-\alpha_m^2/\nu$. For small values of ν, a magnetic field attenuates very quickly independently on the direction of motion of a shell since $\mathrm{sgn}(\gamma)=-\mathrm{sgn}(\nu)$. For $\nu\rightarrow\pm\infty$, $\gamma\rightarrow\mp 0$ and the dynamics of an electromagnetic field is the same as in the case of an ideally conducting shell which was analyzed in Ref. [13].

This completes our analysis of configurations with a cylindrical symmetry. In the following we analyze amplification of electromagnetic fields in spherical geometry.

V. AMPLIFICATION OF MAGNETIC FIELDS BY A SPHERICAL SHOCK WAVE

Consider an amplification of an electromagnetic field by a spherical shock wave. It is known that at the advanced stage of the implosion, the converging spherical shock wave can be described by a self-similar solution (see Ref. [16], Chap. 10, Sec. 107). Assume that after the shock wave front a fluid is transformed into a conducting state with a constant electric conductivity σ while a space before the shock wave front $r<R(t)$ is occupied by a stationary nonconducting fluid. As was pointed out above, the stronger the shock wave the less significant is the variation of the electric conductivity after the shock wave front in comparison with a jump of the electric conductivity at the shock wave front.

According to Ref. [16] the self-similar solution for a fluid velocity after the shock wave is given by the following expression:

$$u(r,t)=D\dot{R}(t)\chi^{(\alpha-1)/\alpha},\quad \chi=\frac{r}{R(t)},\quad D=\frac{u_0(t)}{\dot{R}(t)}, \tag{32}$$

where $R(t)$ is the speed of a shock wave front, $u_0(t)$ is a fluid velocity at the shock wave front, and the power α depends upon the ratio of specific heats c_P/c_V.

Consider an excitation of an electromagnetic field with a magnetic dipole symmetry. This field is determined by a vector potential $\vec{A}=(0,0,A_\varphi)$ where

$$A_\varphi=-a_0\left(\frac{R(t)}{R_0}\right)^\gamma \Phi_\ell(\chi)P_\ell^1(\cos\theta) \tag{33}$$

and $P_\ell^1(cos\ \theta)$ are associated Legendre polynomials.

Expressions for electric field $\vec{E}=(0,0,E_\varphi)$ and a magnetic field $\vec{H}=(H_r,H_\theta,0)$ can be obtained using the following equations:

$$E_\varphi=-\frac{1}{c}\frac{\partial A_\varphi}{\partial t},$$

$$H_r=\frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}+\cot\theta\frac{A_\varphi}{r},$$

$$H_\theta=-\left(\frac{\partial A_\varphi}{\partial r}+\frac{A_\varphi}{r}\right).$$

Then

$$E_\varphi=a_0\beta\left(\gamma\Phi_\ell(\chi)-\chi\frac{\partial\Phi_\ell(\chi)}{\partial\chi}\right)\left(\frac{R(t)}{R_0}\right)^\gamma P_\ell^1(\cos\theta),$$

$$\beta=\frac{\dot{R}}{c}, \tag{34}$$

$$H_r=a_0\frac{\ell(\ell+1)}{R(t)}P_\ell^0(\cos\theta)\left(\frac{R(t)}{R_0}\right)^\gamma\frac{\Phi_\ell(\chi)}{\chi},$$

$$H_\theta=\frac{a_0}{R(t)}P_\ell^1(\cos\theta)\left(\frac{\Phi_\ell(\chi)}{\chi}+\frac{\partial\Phi_\ell(\chi)}{\partial\chi}\right)\left(\frac{R(t)}{R_0}\right)^\gamma,$$

where $\Phi_\ell(\chi)$ is determined by the following equation:

$$\frac{\partial^2\Phi_\ell}{\partial\chi^2}+\frac{2}{\chi}\frac{\partial\Phi_\ell}{\partial\chi}-\frac{\ell(\ell+1)}{\chi^2}\Phi_\ell =\nu_0\left[\left(\gamma+\frac{D}{\chi^{1/\alpha}}\right)\Phi_\ell+(D\chi^{(\alpha-1)/\alpha}-\chi)\frac{\partial\Phi_\ell}{\partial\chi}\right]. \tag{35}$$

Equation (35) is derived from Eqs. (3,34).

Amplification of a magnetic field inside an imploding ideally conducting spherical shell was considered in Ref. [13]. In the latter study we analyzed a more general problem about the eigenmodes of a spherical ideal resonator with radially expanding or converging wall. It was shown that there exist two types of eigenmodes, namely, static eigenmodes which coincide with the solution of equation $\vec{\nabla}^2A_\varphi=0$ when a velocity of a shell tends to zero, and oscillatory eigenmodes which coincide with the eigenmodes of a spherical resonator when a velocity of a shell vanishes (see Ref. [7]). Both eigenmodes are determined by the same boundary condition:

$$\left.\frac{\partial A_\varphi}{\partial t}+\nu\left(\frac{\partial A_\varphi}{\partial r}+\frac{A_\varphi}{r}\right)\right|_{r=R(t)}=0 \tag{36}$$

or $(\gamma+1)\Phi_\ell(1)=0$. (There is a misprint in Ref. [13] in Eq. (26). The correct Eq. (26) is $(\lambda-\ell)\Phi_\ell(1)=0$.) The oscillatory modes satisfy a condition $\Phi_\ell(1)=0$ while the static modes describing an amplification of a magnetic field satisfy a condition $\gamma=-1$. In the latter case with an accuracy of the order of $\beta^2\chi^2<\beta^2$, $\Phi_\ell(\chi)=\chi^\ell$, and it can be easily seen that A_φ is a solution of equation $\hat{\nabla}^2A_\varphi=0$.

In a case with a finite electric conductivity a vector potential $A_\varphi(r,t)$ in the region $\chi<1$ can be written as

$$A_\varphi=-a_0\left(\frac{R(t)}{R_0}\right)^\gamma\chi^\ell P_\ell^1(\cos\theta). \tag{37}$$

In order to solve Eq. (35) analytically we selected $\alpha=1/2$ which is close to the value $\alpha\approx0.7$ for a shock wave (see Ref. [16], Chap. 10, Sec. 107). In this case a solution of Eq. (35) which vanishes when $\chi\to\infty$ can be written as follows:

$$\Phi_\ell(\chi)=\chi^s\Psi\left(\overline{a},\overline{c};-\frac{\nu_0\chi^2}{2}\right),$$

$$s=-\frac{1-\nu_0D}{2}-\sqrt{\frac{(1+\nu_0D)^2}{4}+\ell(\ell+1)}, \tag{38}$$

$$\overline{c}=1-\sqrt{\frac{(1+\nu_0D)^2}{4}+\ell(\ell+1)},\quad \overline{a}=\frac{s-\gamma}{2},$$

where $\Psi(\overline{a},\overline{c};z)$ is determined by Eq. (16). Condition of continuity $\partial(\ell n\Phi)/\partial\chi$ at $\chi=1$ yields a transcendental equation for a parameter $\gamma=\gamma(\nu_0,D,\ell)$:

$$(\ell-s)\Psi\left(\overline{a},\overline{c},-\frac{\nu_0}{2}\right)=\nu_0\overline{a}\Psi\left(\overline{a}+1,\overline{c}+1,-\frac{\nu_0}{2}\right). \tag{39}$$

Since at $\chi\to\infty$, $\Phi_\ell(\chi)\propto\chi^\gamma$, a requirement for the realizability of this model is $\gamma<0$ [see Eq. (6)]. The limiting value $\gamma=-1$ corresponds to $\nu_0D\to\infty$. Thus it can be assumed that in the range of physical realizability of this model $-1\leqslant\gamma<0$, $\nu_0D\gg1$. When the values ℓ are such that condition $\ell^2\ll\nu_0D$ is satisfied, in the leading order of the parameter $\ell^2/(\nu_0D)$

$$s=-1,\quad \overline{c}=-\frac{\nu_0D}{2},\quad \overline{a}=-\frac{1+\gamma}{2}. \tag{40}$$

Then instead of Eq. (39) we have

$$(\ell+1)\Psi\left(\overline{a},\overline{c},-\frac{\nu_0}{2}\right)=\nu_0\overline{a}\Psi\left(\overline{a}+1,\overline{c}+1,-\frac{\nu_0}{2}\right). \tag{41}$$

Since for $\ell=1$ Eqs. (40,41) coincide with Eq. (19) the plots shown in Fig. 1 correspond also to this case. The dependence of the exponent γ in Eq. (37) vs magnetic Reynolds number ν_0 for $\ell=1,2,3,4$ and $D=1$ is shown in Fig. 2. Inspection of these plots shows that an amplification of a magnetic field with a higher value of a parameter ℓ requires larger velocities of implosion.

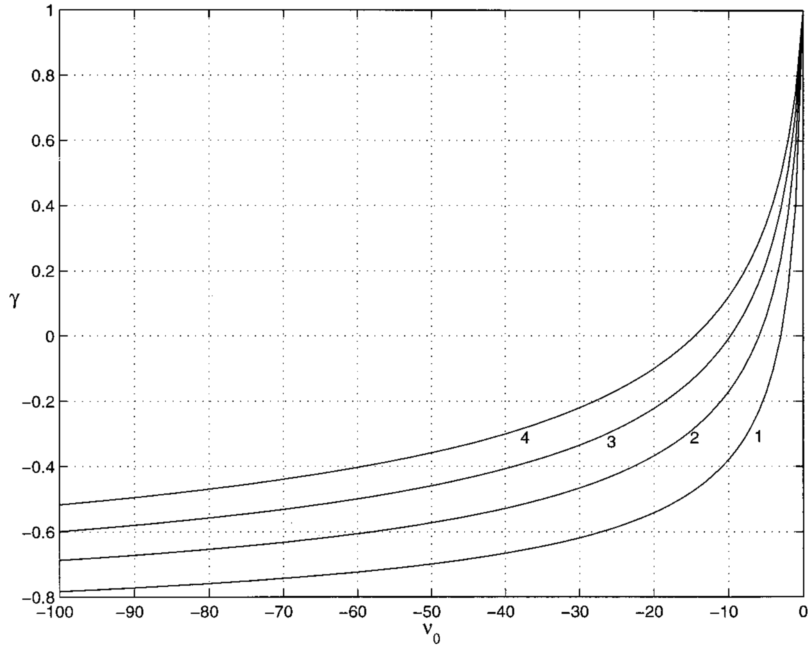

FIG. 2. The dependence of the exponent γ in Eq. (37) vs magnetic Reynolds number ν_0 for various values of a wave number ℓ. $1-\ell=1; 2-\ell=2; 3-\ell=3; 4-\ell=4$.

VI. CONCLUSIONS

We considered various geometries where amplification of the electromagnetic field can occur. Since our goal was to derive analytical results we studied only simple geometries. Using the same approach it is possible to study an amplification of the electromagnetic field in a gap between two infinite conducting plates. Since the solution of the latter problem is only slightly different from the solutions obtained in this study we did not present it here.

It was found that although a condition for an amplification of the electromagnetic field is different for various geometries the sufficient condition for amplification for any geometry is that $\nu_0 \sim 10$. For a conductor with $\sigma_0 \sim 10^{16}\,\mathrm{s}^{-1}$ and $\ell \sim 1$ cm, a material velocity must be of the order of $\dot{l} \sim 10^4$ cm/s. Certainly the above estimate of ν_0 may become somewhat larger if one takes into account heating and the dependence of the electric conductivity upon temperature (see Ref. [12]).

[1] T. Erber and H. G. Latal, Rep. Prog. Phys. **33**, 1069 (1970).
[2] K. Nagayama, Appl. Phys. Lett. **38**, 109 (1981).
[3] E. I. Bichenkov, S. D. Gilev, A. M. Ryabchun, and A. M. Trubachev, J. Appl. Mech. Tech. Phys. **28**, 331 (1987).
[4] Yu. Dolinsky and T. Elperin, Phys. Rev. E **54**, 2994 (1996).
[5] Yu. Dolinsky and T. Elperin, Phys. Rev. E **56**, 3633 (1997).
[6] Yu. Dolinsky and T. Elperin, J. Appl. Phys. **81**, 2935 (1997).
[7] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1984).
[8] *Strong and Ultrastrong Magnetic Fields and Their Applications*, edited by F. Herlach (Springer, Berlin, 1985).
[9] A. Paton and W. Millar, J. Appl. Phys. **35**, 1141 (1964).
[10] V. A. Afonin, Yu. D. Bakulin, and A. V. Luchinskii, J. Appl. Mech. Tech. Phys. **21**, 743 (1980).
[11] A. L. Velikovich, S. M. Golberg, M. A. Liberman, and F. S. Felber, Zh. Eksp. Teor. Fiz. **88**, 445 (1985) [Sov. Phys. JETP **61**, 261 (1985)].
[12] A. L. Velikovich and M. A. Liberman, Zh. Eksp. Teor. Fiz. **89**, 1205 (1985) [Sov. Phys. JETP **62**, 694 (1985)].
[13] Yu. Dolinsky and T. Elperin, J. Appl. Phys. **83**, 5644 (1998).
[14] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, New York, 1964).
[15] E. Jahnke, F. Emde, and F. Lösch, *Special Functions* (B. G. Teubner Verlagsgesellschaft, Stuttgart, 1960).
[16] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).